Alberta Thy xx-99
October 1999

# Holographic Counterterm Actions and Anomalies for Asymptotic AdS and Flat Spaces

Jeongwon Ho

Theoretical Physics Institute, Department of Physics,
University of Alberta, Edmonton, Canada T6G 2J1
E-mail: jwho@phys.ualberta.ca

**Abstract**

Counterterm actions for regularizing gravitational action on non-compact spaces are studied in evaluating the action value on boundaries. It is shown that the boundary action value, so the counterterm action, always can be written as an integration form of intrinsic boundary geometry. We consider some examples and show that our expression of counterterm action reproduces previous results. We discuss about the relationship between counterterm actions for asymptotically AdS and flat. Using this description, we also discuss about the holographic anomaly. For asymptotic AdS spaces, restricting the boundary geometry, we obtain an arbitrary (even) dimensional holographic anomaly. Interestingly, we also show that for an asymptotically flat space, the boundary action value has a logarithmically divergent term, and corresponding conformal anomaly is obtained. According to this anomaly, descriptions and applications are briefly discussed.

# 1 Introduction

There has been a typical problem to define a gravitational action suffering from divergence in a non-compact space. In spite that several prescriptions within the concept of reference space have been suggested so far [1][2][3], those are flawed by the fact although the divergences could be eliminated by choosing an appropriate reference space, it is impossible to embed a boundary with an arbitrary geometry. Another drawback of the reference space method is that different reference spaces are needed for different boundary geometries, so that one cannot define relative energies in a consistent manner.

Recently, a prominent prescription has been suggested [4] in context of AdS/CFT correspondence [5][6][7]. According to the correspondence, the UV divergences of quantum field theory living on the boundary of AdS space are derived from the IR divergences of the bulk theory. So, the bulk action could be regularized by adding local counterterms [6][8]. For asymptotically AdS spaces, this approach gives an elegant expression of counterterm action in the form of the expansion for AdS radius $\ell$ [4][9][10][11]

$$\begin{aligned}\tilde{S} &= -\frac{1}{8\pi G}\int_{\partial X} d^d x\sqrt{-g_0}\left\{\frac{d-1}{\ell}+\frac{\ell}{2(d-2)}R\right.\\ &\left.+\frac{\ell^3}{2(d-2)^2(d-4)}\left(R_{ab}R^{ab}-\frac{d}{4(d-1)}R^2\right)+\cdots\right\}. \qquad (1)\end{aligned}$$

In case of even dimensional boundary, however, one encounters a logarithmic divergent term in evaluating the bulk action functional. What one takes the counterterm action involving this log term to render the action finite may produce problematic results in calculating the boundary stress energy [10]. Even though this appearance of the logarithmic divergent term embarrasses the counterterm subtraction approach to define a finite regularized action, it provides a remarkable consistency check of the AdS/CFT correspondence [6][8]. The conformal anomaly for $d$-dimensional conformal field theories in coupling to background gravity comes from logarithmic UV divergences [12]. Thus, to evaluate the conformal anomaly in this scheme becomes a nontrivial check of the UV-IR connection [13] of the AdS/CFT correspondence.

In more systematic scheme, the counterterm action of Eq.(1) has been also constructed from the Gauss-Codazzi equations through an iterative process [11]. The authors investigated counterterm actions for asymptotically flat (AF) spaces as well. Unfortunately, the procedure adapted for AdS spaces could not be simply generalized on AF spaces because of a mathematical difficulty due to non-linearity of the Gauss-Codazzi equations. Taking an alternative approach, they obtained a counterterm action for AF spaces with $S^{d-n} \times \mathbb{R}^n$ boundary geometries

$$\tilde{S} = -\frac{1}{8\pi G} \int_{\partial X} d^d x \sqrt{-g_0} \sqrt{\frac{R^3}{R^2 - R_{ab} R^{ab}}}. \tag{2}$$

Very recently, a different prescription to construct the counterterm action has been suggested [14]. In the prescription, a length dimensional parameter analogue to the radius of AdS space was defined, so that the counterterm actions for asymptotically flat and AdS spaces could be consistently constructed in the expansion for the defined length parameter.

The authors of Ref.[11] considered an interesting example of spaces with nontrivial boundary geometry that is the $D$-dimensional generalization of the Kerr metric [15] setting the mass parameter to zero. It is the metric of AF space in spheroidal coordinates. They have shown that for $d \geq 6$ the counterterm action in (2) can not eliminate all divergent terms. In the procedure, the full Einstein equations were used in evaluating the action value on the boundary. However, since the counterterm action is given by a surface integral on the boundary, one has to calculate the boundary action value in which the boundary integral still remains. Thus, on action level, only the equation that is obtained by projecting the Einstein equations on the boundary in normal directions, must be used. In fact, for spaces with simple boundary geometries, it does not matter whether another equations, tangential-tangential and tangential-normal projection equations of the Einstein equations, are used or not, because they are trivial or dummy in deriving the counterterm action. However, on nontrivial boundary geometries, what one uses the full Einstein equations is to get an over-constrained boundary action value (BAV). On the above example, we shall show that there are missing terms in the BAV comparing with the evaluation of the Ref.[11], and the missing terms produces logarithmic divergence in even dimensional boundaries. It is very interesting result, because according to the holographic principle [16][17], it can be identified with the UV logarithmic divergences of the boundary theory that is related to the conformal anomaly.

The first step of the process will be given by evaluation of the BAV with only the normal-normal projection equation. Then, we will show that the BAV, so the counterterm action, always can be written by intrinsic geometry of a boundary. From this expression of the counterterm action, we obtain some valuable results. In fact, concerning the variety of the boundary geometries, the expression of the counterterm action in (1) is very elegant one. However, for higher dimensions, its evaluation is not manageable. In this paper, restricting our concern to simple boundary geometry, we obtain the counterterm action available for any $d$-dimensional boundary. From this expression, we derive arbitrary dimensional (holographic) anomaly.

Our paper is organized as follows; In Sect.2, counterterm action is formulated as an integration form. Examples for asymptotic AdS spaces are considered in Sect.3; A counterterm action for asymptotic AdS space with $S^d$ boundary is constructed. From this example, arbitrary dimensional conformal anomaly is obtained. In Sect.4, the relationship between the counterterm actions for asymptotic AdS and flat spaces is discussed. For AF space in spheroidal coordinates, the BAV and counterterm action is evaluated. From this, holographic anomaly is obtained and according to the results, some speculations will be given given. Discussions and summary are contained in Sect.5.

## 2 Holographic Counterterm Actions

$(d+1)$-dimensional gravitational action with cosmological constant $\Lambda = -d(d-1)/(2\ell^2)$ is given by

$$S = \frac{1}{16\pi G}\int_X d^{d+1}x\sqrt{-\hat{G}}\left(\hat{R} + \frac{d(d-1)}{\ell^2}\right) - \frac{1}{8\pi G}\int_{\partial X} d^dx\sqrt{-g}\Theta, \qquad (3)$$

where $g_{ab}$ is boundary metric and $\Theta$ is the trace of extrinsic curvature of $d$-dimensional timelike boundary $\partial X$ defined by $\Theta_{ab} = -g_a^\mu \nabla_\mu n_b$. $\nabla$ denotes the covariant derivative on $(d+1)$-dimensional manifold $X$ and $n^\mu$ is an outward unit normal to the boundary $\partial X$. The boundary term in Eq.(3), so called Gibbons-Hawking term, is required for well defined variational principle.

Our purpose is to add another proper surface integral to the action in (3), so that the action becomes finite in the limit that the boundary is taken to infinity.

According to the counterterm subtraction approach, the integrand of the additional surface integral must be given by the inverse sign of the divergent terms of the BAV. In addition, it must be given by a functional of intrinsic boundary geometry. For the procedure, we take the ADM formulation as a guide line for construction of the counterterm action. As it will seen in the following, the ADM formulation guarantees to extract the intrinsic boundary geometries from the action (3).

To rewrite the action (3) in a canonical form, we first take a metric given by

$$\hat{G}_{\mu\nu}dx^{\mu}dx^{\nu} = N^2 d\rho^2 + g_{ab}dx^a dx^b, \tag{4}$$

where $N^2 = N^2(\rho)$ and $g = g(\rho, x^a)$. On this coordinate system, the unit normal to the boundary is given by $n_\mu = N\delta^\rho_\mu$. Then, following the standard ADM formalism, the canonical form of the action (3) is

$$S = \int_X d^{d+1}x(\pi^{ab}g'_{ab} - N\mathcal{H}_\rho) \equiv \int_X d^{d+1}x\mathcal{L}, \tag{5}$$

where $\pi^{ab} = \delta\mathcal{L}/\delta g'_{ab}$ is the momentum density conjugate to $g_{ab}$ and $'$ denotes the derivative of $\rho$. 'Hamiltonian' density $\mathcal{H}_\rho$ is given by

$$\mathcal{H}_\rho = \frac{16\pi G}{\sqrt{-g}}\left(\frac{\pi^2}{d-1} - \pi_{ab}\pi^{ab}\right) - \frac{\sqrt{-g}}{16\pi G}\left(R + \frac{d(d-1)}{\ell^2}\right), \tag{6}$$

where $R$ is the $d$-dimensional scalar curvature of the boundary. The equation $\mathcal{H}_\rho = 0$ generates reparametrization of space coordinate $\rho$. In fact, this equation one of the Gauss-Codazzi equations that is defined by projecting the Einstein equations on the boundary in normal directions.

Using the constraint equation $\mathcal{H}_\rho = 0$, the BAV evaluated from the action in (3) on the boundary $\rho = \rho_0$ is given by a simple form as

$$\begin{aligned} S_{cl} &= \int_{\partial X} d^d x \left\{\frac{1}{8\pi G}\int^{\rho_0} d\rho N\sqrt{-g}\left(R + \frac{d(d-1)}{\ell^2}\right)\right\} \\ &\equiv \int_{\partial X} d^d x A(x^a;\rho_0). \end{aligned} \tag{7}$$

So, according to the counterterm subtraction approach, regularized action (RA), $S_{RA}$, is defined by

$$S_{RA} \equiv S - \tilde{S}, \tag{8}$$

where the counterterm action $\tilde{S}$ is given by

$$\tilde{S} = -\int_{\partial X} d^d x Div\left(A(x^a;\rho_0)\right) \equiv -\int_{\partial X} d^d x\sqrt{-g_0}\bar{A}^{div}(x^a;\rho_0), \tag{9}$$

where $Div$ means to pick divergent terms after $\rho$-integration and $g_0$ is the induced metric on the boundary.

In fact, the expression for the counterterm action (9) still contains an integration of a space coordinate $\rho$ in the functional $A$ that must be evaluated in case-by-case. However, from the expression of the counterterm action, we find two important things in our investigation. One is that the integrand of the counterterm action in Eq.(9) is to be a functional of the *intrinsic* boundary geometry, because $A(x^a;\rho_0)$ in Eq.(7) is the functional of the intrinsic boundary geometry. (The 'lapse' function $N$ can be eliminated by a redefinition of the space coordinate $\rho$.) In fact, the divergent terms in BAV is given from the Gibbons-Hawking term that is the surface integral of the extrinsic curvature, as well as from the bulk terms. Of course, it has been shown that the divergent terms of the Gibbons-Hawking term can be described by intrinsic boundary geometry in some examples [11]. In the above procedure, it can be seen very clearly. The extrinsic term is canceled by a term extracted from the bulk term, and the whole divergent structure of the BAV in (7) is given by only the bulk term written by the intrinsic boundary geometry.

On the other hand, it must be also noted that in the above procedure, we did not use the full Einstein's equation to obtain the BAV, but only the constraint equation, $\mathcal{H}_\rho = 0$, was used. This means that in the BAV in (7), degrees of freedom inherited on the boundary still remain without fixing it. What one uses another Gauss-Codazzi equations is to lead to an over-constrained BAV. In section 4, it will be seen that this notion leads to a very interesting result.

# 3 AdS Space and Holographic Anomaly

The counterterm action for asymptotic AdS spaces in (1) is useful for various boundary geometries. However, counterterm action for higher dimensional boundary is not easy to evaluate because of its mathematical complexity. If one restricts the boundary geometry to simple one, this complexity properly disappears. As an example, we consider the Euclidean AdS space with $S^d$ boundary described by the line element

$$\hat{G}_{\mu\nu}dx^\mu dx^\nu = \left(1+\frac{r^2}{\ell^2}\right)^{-1} dr^2 + r^2 d\Omega_d^2. \tag{10}$$

This specific example is to give us some valuable results.

The counterterm action for this asymptotic space can be constructed by evaluating the functional $A$ in Eq.(7) for the metric (10)

$$
\begin{aligned}
A(x^a; r_0) &= \frac{1}{8\pi G}\int^{r_0} dr\sqrt{\gamma_d} r^d \left(1+\frac{r^2}{\ell^2}\right)^{-1/2}\left(R+\frac{d(d-1)}{\ell^2}\right) \\
&= -\frac{(d(d-1))^{(d+2)/2}}{16\pi G\ell}\sqrt{\gamma_d}\int^{R_0} dR R^{-(d+2)/2}\left(1+\frac{\ell^2 R}{d(d-1)}\right)^{1/2}, \quad (11)
\end{aligned}
$$

where $R_0$ denotes the scalar curvature on the boundary and $\gamma_d$ is the metric of $d$-dimensional unit sphere. In the second line of Eq.(11), $d(d-1)/r^2 = R$ was used. After some algebraic calculation, we obtain

$$
\begin{aligned}
A(x^a; r_0) &= \frac{2}{d(d-1)}\sqrt{1+\frac{\ell^2 R}{d(d-1)}} \quad (12)\\
&\times \left\{-\frac{d-1}{R^{d/2}} + \sum_{k=1}^{(d-2)/2}\left[\left(-\frac{\ell^2}{d(d-1)}\right)^k \prod_{m=1}^{k}\left(\frac{d-2m+1}{d-2m}\right) R^{-(d-2k)/2}\right]\right\} \\
&- \frac{1}{d}\left(-\frac{\ell^2}{d(d-1)}\right)^{d/2}\prod_{k=1}^{(d-2)/2}\left(\frac{d-2k-1}{d-2k}\right)\ln\frac{\sqrt{1+\ell^2 R/(d(d-1))}-1}{\sqrt{1+\ell^2 R/(d(d-1))}+1}
\end{aligned}
$$

in even of $d$ and

$$
\begin{aligned}
A(x^a; r_0) &= -\frac{2}{d-1}\left(1+\frac{\ell^2 R}{d(d-1)}\right)^{3/2}\left(-\frac{\ell^2}{d(d-1)}\right)^{(d-5)/2} \\
&\times \sum_{k=0}^{(d-3)/2}\prod_{m=0}^{k}\left(\frac{d-2m-1}{d-2m}\right) \quad (13)
\end{aligned}
$$

in odd of $d$. In E's.(12), (13) and here after, we drop the subscript '0' of the scalar curvature for simplicity. Then, the *arbitrary* dimensional counterterm action for AdS spaces with $S^d$ boundary is given by a polynomial in the boundary scalar curvature $R$ as follows

$$
\begin{aligned}
\tilde{S} &= \frac{1}{8\pi G}\int_{\partial X} d^d x\sqrt{g_0}\left(\frac{d-1}{\ell}+\frac{\ell}{2(d-2)}R-\frac{\ell^3}{8d(d-1)(d-4)}R^2\right. \\
&\left.+\frac{\ell^5}{16(d(d-1))^2(d-6)}R^3+\cdots\right), \quad (14)
\end{aligned}
$$

where the terms in parenthesis of Eq.(14) are terminated by

$$
\frac{1}{2}(-1)^{(d+2)/2}\prod_{k=1}^{(d+2)/2}\left(\frac{2k-3}{2k}\right)\frac{\ell^{d+1}}{(d(d-1))^{d/2}}R^{(d+2)/2}, \quad (15)
$$

in the case of $d = even$, and

$$(-1)^{(d+1)/2}\prod_{k=1}^{(d-1)/2}\left(\frac{2k-3}{2k}\right)\frac{\ell^{d-2}}{(d(d-1))^{(d-3)/2}}R^{(d-1)/2}, \tag{16}$$

in odd $d$ case.

After simple algebraic calculation using the relation $R_{ab}R^{ab} = R^2/d$, it can be shown that the counterterm action in (14) is equivalent to the Eq.(1). In other words, the counterterm action in (1) can be written by a polynomial in the boundary scalar curvature $R$ for the AdS spaces with $S^d$ boundary and in the case, $d$-dimensional counterterms are determined by the terms in Eqs.(15) or (16).

It must be also noted that since we take the counterterm action (14) as a polynomial in the $d$-dimensional scalar curvature $R$, in $d = even$ case, the counterterm action of Eq.(14) fails on eliminating all divergent terms appearing in the BAV, instead the RA contains a logarithmically divergent term

$$\frac{1}{16\pi G}\int d^dx\sqrt{g_0}(-1)^{d/2}\prod_{k=1}^{d/2}\left(\frac{2k-3}{2k}\right)\frac{\ell^{d-1}}{(d(d-1))^{(d-2)/2}}R^{d/2}\ln R. \tag{17}$$

It has been already understood in the context of the AdS/CFT correspondence [6][8]; The regularization of BAV by introducing local counterterms may break conformal invariance and RA is left with a conformal anomaly for boundary CFT. According to this prescription, we derive (holographic) conformal anomaly for which the dual CFT is coupled to the background gravity with $S^d$ boundary. Considering a scale transformation $\delta r = r\delta\epsilon$ for an infinitesimal constant parameter $\delta\epsilon$, the conformal invariance is broken by the conformal anomaly $\mathcal{A}$

$$\mathcal{A} = -\frac{1}{8\pi G}(-1)^{d/2}\prod_{k=1}^{d/2}\left(\frac{2k-3}{2k}\right)\frac{\ell^{d-1}}{(d(d-1))^{(d-2)/2}}R^{d/2}. \tag{18}$$

The conformal anomaly in arbitrary dimensions has been given in geometric description [18]. Restricting the CFT in background $S^d$ geometry, Eq.(18) is an alternative expression of the conformal anomaly in arbitrary dimensions.

For $S^2$ boundary, the Eq.(18) recovers well known result

$$\mathcal{A}_{d=2} = -\frac{\ell}{16\pi G}R. \tag{19}$$

Comparing the $(1+1)$-dimensional anomaly on a surface of radius $\ell$, $-1/(8\pi G\ell) = -c/(12\pi\ell^2)$, the central charge $c$ becomes $3\ell/(2G)$. From the Eq.(18) for $d=4$, we find that the conformal anomaly agrees with that of Ref.[8]

$$\mathcal{A}_{d=4} = \frac{\ell^3}{768\pi G}R^2 = \frac{\ell^3}{8\pi G}\left(-\frac{1}{8}R_{ab}R^{ab} + \frac{1}{24}R^2\right). \qquad (20)$$

The conformal anomaly for $\mathcal{N}=4$ super Yang-Mill theory on $S^4$ is $3N^2/(8\pi^2\ell^4)$. Comparing with the anomaly on this boundary from the Eq.(20), $3/(16\pi G\ell)$, we obtain the expected result

$$N^2 = \frac{\pi\ell^3}{2G}, \qquad (21)$$

where $N$ is the rank of the gauge group of the dual $\mathcal{N}=4$ supersymmetric $d=4$ $SU(N)$ YM theory. At last, it can be seen that for six dimensional boundary, the anomaly in (18) is equivalent to that of [8]

$$\begin{aligned}\mathcal{A}_{d=6} &= -\frac{\ell^5}{115200\pi G}R^3 \\ &= -\frac{1}{16\pi G}\left(\frac{\ell^5}{64}\right)\left(\frac{1}{2}RR_{ab}R^{ab} - \frac{3}{15}R^3 - R^{ab}R_{acbd}R^{cd}\right. \\ &\quad \left. + \frac{1}{5}R^{ab}D_aD_bR - \frac{1}{2}R^{ab}\Box R_{ab} + \frac{1}{20}R\Box R\right). \qquad (22)\end{aligned}$$

In fact, since we are concerned about $S^6$ boundary, the terms in third line including derivatives vanish. On the other hand, Eq.(22) can be verified by considering the central charge of $N$ coincident M5-branes in the large $N$ limit. It has been shown that the central charge is proportional to $N^3$ [19]. So, the anomaly on $S^6$ boundary with radius $\ell$, $15/(64\pi G\ell)$, is proportional to $N^3/(\pi^4\ell^6)$. Thus, we find [5]

$$N^3 \sim \frac{\pi^3\ell^5}{G}. \qquad (23)$$

Before ending of this section, it is useful on following process to consider another Euclidean AdS space with different boundary geometry, $S^{d-1}\times S^1$,

$$ds^2 = \left(1+\frac{r^2}{\ell^2}\right)d\tau^2 + \left(1+\frac{r^2}{\ell^2}\right)^{-1}dr^2 + r^2 d\Omega_{d-1}^2 . \qquad (24)$$

On this metric, the functional $A$ in (7) is given by

$$A(x^a;r_0) = \frac{\sqrt{\gamma_{d-1}}}{8\pi G}\left(\frac{d-1}{\ell^2}\right)\left(\frac{(d-1)(d-2)}{R}\right)^{d/2}\left(1+\frac{\ell^2 R}{(d-1)(d-2)}\right). \qquad (25)$$

Since all terms in expanding of Eq.(25) are divergent for $d > 2$, the counterterm action is just the negative of $S_{cl}$ in (7) [11]

$$\tilde{S} = \frac{1}{8\pi G}\int_{\partial X} d^d x\sqrt{g_0}\left(\frac{d-1}{\ell}\right)\left(1+\frac{\ell^2 R}{(d-1)(d-2)}\right)^{1/2}. \tag{26}$$

It could be shown that using $R_{ab}R^{ab} = R^2/(d-1)$ and expanding for $\ell$, the counterterm action in (26) is equivalent to Eq.(1). However, it must be noted that while the counterterm action for $S^d$ boundary is given by a finite sum of the series in (1), for $S^{d-1}\times S^1$ boundary it is given by an infinite sum. As mentioned in Ref.[11], in the process the divergent factors $1/(d-4)$, $1/(d-6)$, $\cdots$ in (1) are canceled. Thus, while conformal invariance of the RA for $S^d$ boundary is broken by the anomaly in (18), that for the $S^{d-1}\times S^1$ is still conformal invariant.

# 4 AF Space and Holographic Anomaly

Now, consider the counterterm action for asymptotic flat spaces. Those are simply obtained by taking the limit of $\ell \to \infty$ on the functional $A$'s

$$\tilde{S} = -\frac{1}{8\pi G}\int_{\partial X} d^d x\sqrt{-g_0}\sqrt{\frac{dR}{d-1}} \tag{27}$$

in (12) and (13), and

$$\tilde{S} = -\frac{1}{8\pi G}\int_{\partial X} d^d x\sqrt{-g_0}\sqrt{\frac{d-1}{d-2}R} \tag{28}$$

in (25). In Eqs.(27) and (28), the counterterm actions were written in the Lorentzian signature. In more useful expression, the counterterm actions in (27) and (28) can be written by [11]

$$\tilde{S} = -\frac{1}{8\pi G}\int_{\partial X} d^d x\sqrt{-g_0}\sqrt{\frac{R^3}{R^2 - R_{ab}R^{ab}}}. \tag{29}$$

In fact, the counterterm action in (29) is somewhat general expression which is available for the AF spaces with $S^{d-n}\times\mathbb{R}^n$ boundary geometries described by the metric

$$\hat{G}_{\mu\nu}dx^\mu dx^\nu = (-dt^2 + dx_1^2 + \cdots + dx_{n-1}^2) + dr^2 + r^2 d\Omega_{d-n}^2. \tag{30}$$

The functional $A$ in (7) for AF spaces becomes

$$A(x^a;\rho_0) = \frac{1}{8\pi G}\int^{\rho_0} d\rho N\sqrt{-g}R, \tag{31}$$

and then the counterterm action is

$$\begin{aligned}\tilde{S} &= -\frac{1}{8\pi G}\int_{\partial X} d^dx\sqrt{-g_0}\sqrt{\frac{d-n}{d-n-1}R} \\ &= -\frac{1}{8\pi G}\int_{\partial X} d^dx\sqrt{-g_0}\sqrt{\frac{R^3}{R^2-R_{ab}R^{ab}}}. \end{aligned} \tag{32}$$

In the above evaluation, it must be noted that the counterterm action for AF spaces is not a limitation of the counterterm action for the asymptotic AdS spaces. In other words, the counterterm action for the two different asymptotic spaces (flat and AdS) are derived by taking different limitations on the BAV in (7) respectively. So, there is not any reasonable relationship between themselves without passing through the BAV. (In Ref.[14], the author has taken a new expanding parameter that varies on different boundary geometry.)

Now, consider an AF space in spheroidal coordinates that was investigated in Ref.[11]

$$\hat{G}_{\mu\nu}dx^\mu dx^\nu = -dt^2 + \frac{\rho^2}{r^2+a^2}dr^2 + \rho^2 d\theta^2 + \sin^2\theta(r^2+a^2)d\phi^2 + r^2\cos^2\theta d\Omega^2_{d-3}. \tag{33}$$

This space can be obtained by setting the mass to zero in higher dimensional Kerr metric [15]

$$\begin{aligned}\hat{G}_{\mu\nu}dx^\mu dx^\nu &= -\frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\phi\right)^2 + \frac{\sin^2\theta}{\rho^2}\left(adt-(r^2+a^2)d\phi\right)^2 \\ &\quad + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + r^2\cos^2\theta d\Omega^2_{d-3}, \end{aligned} \tag{34}$$

where $\rho^2 = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - 2mr + a^2$, and $m$ and $a$ are the black hole mass and the angular momentum per unit mass, respectively. It is important that the metric in (33) does not describe the asymptotic spacetime of the Kerr black hole in (34). Because, in the process, one is to meet a naked singularity. It is just the flat spacetime metric of $n = 1$ in Eq.(30) written in spheroidal coordinates.

As shown in Ref.[11], the counterterm action (29) for the metric (33) can not properly eliminate the divergent terms appearing in BAV. Now, using the Eq.(31), we calculate the counterterm action for the metric in (33). The $d$-dimensional scalar curvature $R$ on the metric is

$$\begin{aligned}R &= \frac{2a^2\left((d-3)\sin^2\theta - \cos^2\theta\right)}{(r^2+a^2\cos^2\theta)^2} + \frac{(d-3)(d-4)}{r^2\cos^2\theta} \\ &\quad + \frac{(2(2d-5)-(d-3)(d-4)\tan^2\theta)}{r^2+a^2\cos^2\theta}. \end{aligned} \tag{35}$$

Inserting the scalar curvature into Eq.(31), and evaluating the functional $A(x^a; r_0)$, then

$$
\begin{aligned}
A(x^a; r_0) &= \frac{\sqrt{\gamma_{d-3}}}{8\pi G} \int^{r_0} dr r^{d-3} \left[ \frac{2a^2 \left( (d-3)\sin^2\theta - \cos^2\theta \right)}{r^2 + a^2\cos^2\theta} \right. \\
&\quad \left. +(d-1)(d-2) + (d-3)(d-4)\frac{a^2}{r^2} \right] \sin\theta \cos^{d-3}\theta \\
&= \frac{\sqrt{\gamma_{d-3}}}{8\pi G} r_0^{d-2} \left[ (d-1) + \left( d - 3 + \frac{2((d-3)\sin^2\theta - \cos^2\theta)}{d-4} \right) \frac{a^2}{r_0^2} \right. \\
&\quad \left. - \frac{2\cos^2\theta((d-3)\sin^2\theta - \cos^2\theta)}{d-6} \frac{a^4}{r_0^4} + \cdots \right] \sin\theta \cos^{d-3}\theta, \qquad (36)
\end{aligned}
$$

where the divergence terms in the bracket are terminated by

$$
2a^2 \left( (d-3)\sin^2\theta - \cos^2\theta \right) (-a^2\cos^2\theta)^{(d-4)/2} r_0^{-(d-2)} \ln r_0 \qquad (37)
$$

in $d = \textit{even}$ case, and

$$
2a^2 \left( (d-3)\sin^2\theta - \cos^2\theta \right) (-a^2\cos^2\theta)^{(d-5)/2} r_0^{-(d-3)} \qquad (38)
$$

in $d = \textit{odd}$ case, respectively. Thus, the counterterm action is

$$
\begin{aligned}
\tilde{S} &= -\frac{1}{8\pi G} \int_{\partial X} d^d x \sqrt{-g_0} \left[ (d-1) + \left( d - 3 + \frac{2((d-3)\sin^2\theta - \cos^2\theta)}{d-4} \right) \frac{a^2}{r_0^2} \right. \\
&\quad \left. - \frac{2\cos^2\theta((d-3)\sin^2\theta - \cos^2\theta)}{d-6} \frac{a^4}{r_0^4} + \cdots \right] \frac{r_0}{\sqrt{(r_0^2 + a^2\cos^2\theta)(r_0^2 + a^2)}}. \qquad (39)
\end{aligned}
$$

Of course, since our procedure to determine the counterterm action is just to take the divergent terms of the BAV and to impose inverse sign, the counterterm action of Eq.(39) exactly cancels the divergent terms in BAV for arbitrary dimensions.

In order to compare with previous evaluation of BAV in Ref.[11], rewriting the divergent terms in the bracket of Eq.(36)

$$
\begin{aligned}
&(d-1) + \left( d - 2 - \cos^2\theta + \frac{d\sin^2\theta - 2}{d-4} \right) \frac{a^2}{r_0^2} \\
&+ \left( (\cos^4\theta - \cos^2\theta) + \frac{\cos^2\theta(\cos^2\theta - d\sin^2\theta)}{d-6} \right) \frac{a^4}{r_0^4} + \cdots, \qquad (40)
\end{aligned}
$$

we find some missing terms, i.e., $r_0^{-2}/(d-4)$, $r_0^{-4}/(d-6), \cdots$. As mentioned in Section 1, this additional divergent terms are mathematically originated from the bulk term. The authors of Ref.[11] imposed the full Einstein equation, $\hat{R} = 0$, to the bulk part of the action (3). However, it leads to over-constrained BAV in the case of nontrivial boundary geometries, e.g., the metric in (33). As seen in the section 2, the BAV is obtained by using only the normal-normal projection equation of the Gauss-Codazzi equations, $\mathcal{H}_\rho = 0$.

On the other hand, there is another interesting outcome in the BAV (36); In the case of even dimensional boundaries of $d \geq 4$, it contains a logarithmic divergent term. So, the conformal invariance of the RA is broken by a conformal anomaly

$$\mathcal{A}^{flat} = \frac{r_0^{d-1}}{4\pi G}\left[\frac{a^2(-a^2\cos^2\theta)^{(d-4)/2}\left((d-3)\sin^2\theta - \cos^2\theta\right)}{r_0^{2(d-2)}\sqrt{(r_0^2 + a^2\cos^2\theta)(r_0^2 + a^2)}}\right]. \tag{41}$$

We can see that setting $a = 0$, the anomaly (41) vanishes. This means that the anomaly is due to a nontrivial slicing of the space. In the metric (33), if we take the limit of $r_0 >> a$, then, the metric becomes

$$\hat{G}_{\mu\nu}dx^\mu dx^\nu = -dt^2 + dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 + r^2\cos^2\theta d\Omega^2_{d-3}. \tag{42}$$

This metric describes the asymptotic space of the black hole (34). For this asymptotic spacetime, the anomaly (41) manifestly disappears. However, as mentioned above, the constant $a$ in (33) cannot be interpreted by the angular momentum of the black hole. Then, the metric (33) does not describe the (quasi-) asymptotic spacetime of the black hole. The constant $a$ is a proper characteristic parameter of the metric (33). Thus, taking the asymptotic spacetime of the metric (33), the anomaly (41) is inevitable.

The appearance of the anomaly in (41) is very interesting. Because, in holographic sense, if one finds a CFT corresponding to the anomaly (41), the gravity theory could be identified with the CFT. As an example, consider five-dimensional spacetime of (33). Then, the conformal anomaly, $\mathcal{A}^{flat}_{d=4}$ becomes

$$\mathcal{A}^{flat}_{d=4} = \frac{r_0^3}{4\pi G}(\sin^2\theta - \cos^2\theta)\left(\frac{a^2}{r_0^6}\right)\left(1 + \frac{a^2\cos^2\theta}{r_0^2}\right)^{-1/2}\left(1 + \frac{a^2}{r_0^2}\right)^{-1/2}. \tag{43}$$

We find that up to leading order, the anomaly can be written by a boundary curvature term as

$$\mathcal{A}^{flat}_{d=4} \sim \frac{r_0^3}{4\pi G}\left(-\frac{1}{40}\Box R + \mathcal{O}\left(\frac{a^4}{r_0^8}\right)\right). \tag{44}$$

Comparing with the case of the AdS space (20), we see that as expected, $r_0$ plays the same role of $\ell$. However, as well known, the $\Box R$ term can be eliminated by introducing another local counterterm. Unfortunately, we do not find an anomaly of a CFT corresponding to the anomaly subtracting the $\Box R$ term in (43).

# 5 Summary and Discussions

The counterterm subtracting method to define a finite gravitational action on non-compact spacetime has been speculated in a direct scheme. It has been shown that the BAV in (7) can be always written by a surface integral of a functional of the intrinsic geometric terms of the boundary. So, the counterterm action constructed from the BAV becomes naturally a surface integral of a intrinsic functional of the boundary geometry. On the other hand, restricting the boundary geometry as $S^d$, we have obtained an expression of the counterterm action available for arbitrary dimensional spaces in (14), (15), and (16). According to this expression, the arbitrary dimensional conformal anomaly has been obtained.

We have also considered the relationship between the counterterm actions for asymptotic flat and AdS spaces. In this concern, it has been shown that counterterm actions for the two different asymptotic spaces are derived by taking different limitations in the BAV in (7). So, there is not any reasonable relationship between themselves without passing through the BAV. Recently, Solodukhin [14] has shown that introducing a new expansion parameter instead of $\ell$, the counterterm actions could be constructed in a consistent manner. This newly defined parameter varies on different boundary geometries and becomes $\ell$ on asymptotic AdS spaces.

The expression of the counterterm action in (9) is not an elegant one. But it is helpful to evade from a trap of over-constrained BAV. We have shown that for a nontrivial boundary geometry, what one use the full Einstein equation may lead to the over-constrained BAV. From this observation, it has been shown that for the asymptotic spacetime of the metric (33), RA has a logarithmic divergence term in an even dimensional boundary. Thus, the RA is broken by the anomaly (41). It was discussed that the anomaly is inevitable in taking the asymptotic space of the metric (33). However, the corresponding CFT could not be found.

Recently, flat-space S-matrix has been studied on the large radius limit in the AdS/CFT correspondence [20][21]. This approach has been suffered from a nonlocal holographic mapping. According to the investigation in this paper, it appears that the investigation of the flat-space scattering could be understood on alternative holographic mapping that is different from the AdS/CFT. In order words, there might be an alternative corresponding chain in which a gravity theory on asymptotically flat is identified with a CFT.

**Acknowledgments:** I thank V. Frolov, A. Zelnikov, and G. Yuri for helpful discussions. This work was supported in part by Korea Science and Engineering Foundation and in part by National Science and Engineering Research Council of Canada.

# References

[1] G.W. Gibbons and S.W. Hawking, Phys. Rev. D**15** (1977) 2752.

[2] J.D. Brown and J.W. York Jr, Phys. Rev. D**47** (1993) 1407.

[3] G.T. Horowitz and S.W. Hawking, Class. Quantum Grav. **13** (1996) 1487.

[4] V. Balasubramanian and P. Kraus, *'A Stress Tensor for Anti-de Sitter Gravity'*, hep-th/9902121.

[5] J. Maldacena, Adv. Theor. Math. Phys. **02** (1998) 231.

[6] E. Witten, Adv. Theor. Math. Phys. **02** (1998) 253.

[7] S.S. Gubser, I.R. Klebanov, and A.M. Polyakov, Phys. Lett. B**428** (1998) 105.

[8] M. Henningson and K. Skenderis, JHEP **07** (1998) 023.

[9] S. Hyun, W.T. Kim, and J. Lee, Phys. Rev. D**59** (1999) 084020.

[10] R. Emparan, C.V. Johnson, and R.C. Myers, *'Surface Terms as Counterterms in the AdS/CFT Correspondence'*, hep-th/9903238.

[11] P. Kraus, F. Larsen, and R. Siebelink, *'The Gravitational Action in Asymptotically AdS and Flat Spacetimes'*, hep-th/9906127.

[12] N.D. Birrell and P.C.W. Davies, *‘Quantum Fields in Curved Space’*, Cambridge University Press, Cambridge (1982).

[13] L. Susskind and E. Witten, *‘The Holographic Bound in Anti-de Sitter Space’*, hep-th/9805114.

[14] S.N. Solodukhin, *‘How to make the gravitational action on non-compact space finite’*, hep-th/9909197.

[15] R.C. Myers and M.J. Perry, Ann. Phys. **172** (1986) 304.

[16] G. ’t Hooft, *‘Dimensional Reduction in Quantum Gravity’*, in *Salamfest* (1993) p.284, gr-qc/9310026.

[17] L. Susskind, J. Math. Phys. **36** (1995) 6377.

[18] S. Deser and A. Schwimmer, Phys. Lett. B**309** (1993) 279.

[19] S.S. Gubser and I.R. Klebanov, Phys. Lett. B**413** (1997) 41.

[20] L. Susskind, *Holography in the Flat Space Limit’*, hep-th/9901079.

[21] S.B. Giddings, *Flat-Space Scattering and Bulk Locality in the AdS/CFT Correspondence’*, hep-th/9907129.